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LETTER TO THE EDITOR

Microscopical analysis of the non-dissipative force on a line vortex in a superconductor

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Abstract. A microscopical analysis of the non-dissipative force F_{nd} acting on a line vortex in a type-II superconductor at $T = 0$ is given. All the work presented assumes a charged BCS superconductor. We first examine the Berry phase induced in the BCS superconducting ground state by movement of the vortex and show how this phase enters into the hydrodynamic action S_{hyd} of the superconducting condensate. Appropriate variation of S_{hyd} gives F_{nd} and variation of the Berry phase term is seen to contribute the Magnus or lift force of classical hydrodynamics to F_{nd}. This analysis, based on the BCS ground state of a *charged* superconductor, confirms in detail the arguments of Ao and Thouless within the context of the BCS model. Our Berry phase, in the limit $e \to 0$, is seen to reproduce the Berry phase determined by these authors for a *neutral* superfluid. We also provide a second, *independent*, determination of F_{nd} through a microscopic derivation of the continuity equation for the condensate linear momentum. This equation yields the acceleration equation for the superflow and shows that the vortex acts as a sink for the condensate linear momentum. The rate at which momentum is lost to the vortex determines F_{nd} in this second approach and the result obtained agrees identically with the previous Berry phase calculation. The Magnus force contribution to F_{nd} is seen in both calculations to be a consequence of the vortex topology and motion.

Already, in the phenomenological/macroscopic models of vortex dynamics in type-II superconductors due to Bardeen and Stephen (BS) and Nozières and Vinen (NV) [1], the form of the non-dissipative force F_{nd} acting on the vortex is controversial. This force is the result of the vortex's interaction with an applied magnetic field H_{ext}, an electric field E due to the vortex motion, and the surrounding condensate of superconducting electrons. The disagreement centres on whether the vortex feels the lift or Magnus force of classical hydrodynamics as a consequence of its motion through the superconducting condensate. In the BS model, the non-dissipative force is due strictly to the Lorentz force $\rho_s h\omega(v_s \times \hat{z})/2$, while in the NV model, the Lorentz force is supplemented by the Magnus force $-\rho_s m K v_s \times \hat{z}$ [2]. In a very interesting paper, Ao and Thouless [3] have returned to this controversy arguing that the correct form for F_{nd} is the NV form, and that the Magnus force contribution to it is a manifestation of a Berry phase induced in the many-body ground state due to the vortex motion. They provide a calculation for a neutral superfluid and argue that the same scenario will also apply for a charged superconductor. Given that the BCS model of superconductivity provides a highly successful microscopic description of the dynamics of a *charged* superconductor, it would be very interesting to see whether or not F_{nd} can be determined using this model of a charged superconductor (together with the starting assumptions common to BS and NV, see below). In this letter we report the results of two such calculations. A detailed presentation and discussion of these calculations will be reported elsewhere [4]. In the first calculation we determine F_{nd} by working with the

BCS superconducting ground state in the case where a vortex is present. This state is first constructed and the Berry phase induced in it by the vortex motion is determined. We *then show how this Berry phase enters into the action describing the hydrodynamic degrees* of freedom of the superconducting condensate. Variation of this action with respect to the vortex trajectory gives $\boldsymbol{F}_{\text{nd}}$ and the result found is seen to take the NV form. In the *second calculation we give a microscopic derivation of the acceleration equation for the* superflow. Together with the expected contributions off the vortex due to spatial variation of the chemical potential, and the electric and magnetic fields present, we also find a *singular term arising from the vortex topology which describes the disappearance of linear* momentum into the vortex. The rate at which this momentum is disappearing gives $\boldsymbol{F}_{\text{nd}}$ and is found to agree identically with the result of the Berry phase calculation. We stress *that the two calculations are independent of each other, and each shows that the Magnus* force (contribution to $\boldsymbol{F}_{\text{nd}}$) is a consequence of the vortex topology and motion, exactly as it is in classical hydrodynamics.

We make use of the Bogoliubov equation to treat the superconducting dynamics. The gap function takes the form $\Delta(\boldsymbol{r}) = \Delta_0(\boldsymbol{r})\exp[-\mathrm{i}\theta]$ in the presence of a line vortex with winding number $\omega = -1$ (in cylindrical coordinates (r, θ, z) centred on the vortex). As in the models of BS and NV, we: (i) assume $T = 0$; (ii) will approximate the non-local character of BCS superconductivity by a local dynamics; (iii) assume $H_{c_1} < H_{\text{ext}} \ll H_{c_2}$ so that vortex–vortex interactions can be ignored and attention can focus on a single vortex; (iv) assume a clean type-II superconductor so that pinning effects can be ignored; and, (v) set $\hbar = m = c = 1$ unless otherwise stated. The solutions of the Bogoliubov equation in the presence of a line vortex are well known [5] and can have positive and negative energies relative to the Fermi energy. The superconducting ground state is constructed by occupation of the negative-energy states. The charge conjugation degree of freedom for the two-component Nambu quasi-particle (NQP) is labelled $2s_z$, and the operator that *creates* a negative-energy NQP is $\gamma_{n\downarrow}$ (where n labels the energy spectrum). Thus, the ground state in the presence of a vortex is

$$|\mathrm{BCS}\rangle = \prod_n \gamma_{n\downarrow}|0\rangle. \tag{1}$$

$\gamma_{n\downarrow}$ depends linearly on the (complex conjugate) of the components of the solutions of the Bogoliubov equation (u_n, v_n) [5]. Adiabatic motion of the vortex generates a Berry phase [6] ϕ_n in the solutions (u_n, v_n). Consequently, $\gamma_{n\downarrow}$ inherits the phase $-\phi_n$ which, from equation (1), causes the ground state to develop the Berry phase $\Gamma = -\sum_n \phi_n$. Because the electrons are electrically charged, one must use the gauge-invariant form of the Berry phase [7]

$$\phi_n(t) = \int_0^t \mathrm{d}\tau\, \langle E_n|\, \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\tau} + \frac{e}{\hbar}A_0(\tau)\,|E_n\rangle.$$

$\phi_n(t)$ is calculated using the solutions of [5], from which one can then obtain the ground-state Berry phase Γ. One finds

$$\Gamma = \int \mathrm{d}\tau\, \mathrm{d}^2x\, \rho_s \left(\frac{1}{2}\dot{\boldsymbol{r}}_0 \cdot \nabla_{\boldsymbol{r}_0}\theta - \frac{e}{\hbar}A_0\right) \tag{2}$$

where $\boldsymbol{r}_0$ is the vortex trajectory, and we work per unit length of the vortex. We see that our result reproduces the Berry phase obtained in [3] for a neutral superfluid in the limit where $e \to 0$.

We now show how the ground-state Berry phase Γ enters into the action describing the hydrodynamic degrees of freedom of the condensate. We begin with the vacuum-to-vacuum transition amplitude for the system of electrons which can be written as a path integral quadratic in the fermion fields via a Hubbard–Stratonovich transformation

$$W = \int \mathcal{D}[\Delta]\,\mathcal{D}[\Delta^*]\,\langle \text{vac}; \Delta(t = T)|U_\delta(T, 0)|\text{vac}; \Delta(0)\rangle.$$

Here $U_\Delta(T, 0) = \mathcal{T}(\exp[-\mathrm{i}\int_0^T \mathrm{d}\tau\, H_{\text{eff}}])$; $H_{\text{eff}} = H_{\text{F}} + L_{\text{em}} + L_{\text{c}}$; H_{F} is the usual BCS Hamiltonian in the presence of a 4-potential $(A_0, \boldsymbol{A})$; L_{em} is the Lagrangian for the induced electric and magnetic fields $(\boldsymbol{E}, \boldsymbol{H} - \boldsymbol{H}_{\text{ext}})$; and L_{c} is the condensation Lagrangian with density $|\Delta|^2/2g$. The action for the condensate $S = S_0 + S_{\text{hyd}}$ is given by

$$\mathrm{e}^{-\mathrm{i}(S_0+S_{\text{hyd}})} = \langle \text{vac}; \Delta(T)|U_\Delta(T, 0)|\text{vac}; \Delta(0)\rangle. \tag{3}$$

S_0 is the action for the bulk degrees of freedom of the condensate; S_{hyd} is the action for the hydrodynamic degrees of freedom; and terms in S containing derivatives of the gap function higher than second order are suppressed. By factoring $U_\Delta(T, 0)$ in equation (3) into a sequence of infinitesimal propagations, and appropriately inserting complete sets of instantaneous energy eigenkets $\{|E_n(t_k)\rangle\}$, evaluation of the matrix element in equation (3) boils down to consideration of propagation over an infinitesimal time interval. Spatial translational invariance, which follows from the assumed absence of pinning sites, ensures that $|\text{vac}; \Delta(0)\rangle$ evolves into the instantaneous ground state $|\text{BCS}(t)\rangle$ of $H_{\text{eff}}(t)$, so the relevant matrix element is $\langle\text{BCS}(t+\epsilon)|U_{\Delta(t)}(t+\epsilon, t)|\text{BCS}(t)\rangle$. One finds [4]

$$\langle\text{BCS}(t+\epsilon)|U_{\Delta(t)}(t+\epsilon, t)|\text{BCS}(t)\rangle = \mathrm{e}^{\mathrm{i}\Gamma\epsilon}\langle\text{BCS}(t)|\mathrm{e}^{-\mathrm{i}H_{\text{eff}}(t)\epsilon}|\text{BCS}(t)\rangle \tag{4}$$

where Γ is the Berry phase developed in $|\text{BCS}(t)\rangle$ due to the vortex motion. The remaining matrix element on the RHS of equation (4) can be evaluated [8]; and the contribution from all infinitesimal time intervals summed. This yields the following result for the hydrodynamic action:

$$S_{\text{hyd}} = \int \mathrm{d}\tau \left[-\hbar\Gamma + \int \mathrm{d}^2x \left[\frac{m\rho_s}{2}v_s^2 + N(0)\tilde{A}_0{}^2 + \frac{1}{8\pi}\left\{(\boldsymbol{H} - \boldsymbol{H}_{\text{ext}})^2 - \boldsymbol{E}^2\right\}\right]\right]$$

in which the ground-state Berry phase Γ appears as a consequence of the adiabatic motion of the vortex. Here $\boldsymbol{v}_s = -(\hbar/2m)[\boldsymbol{\nabla}\phi + (2e\boldsymbol{A})/(\hbar c)]$; ϕ is the gap phase; $N(0)$ is the electron density of states at the Fermi level; $\tilde{A}_0 = eA_0 + (\hbar/2)\,\partial_t\phi$; and $\hbar$, m and c have been re-instated. Appropriate to the scenario of an external current passing through a thin superconducting film in the flux-flow regime, we assume that the superflow is a combination of an applied superflow $\boldsymbol{v} = (\hbar/2m)\,\boldsymbol{\nabla}\beta$ and one that circulates about the moving vortex with velocity $\boldsymbol{v}_{\text{circ}} = -(\hbar/2m)\,\boldsymbol{\nabla}\theta$. The terms in S_{hyd} linear in $\boldsymbol{\nabla}_{r_0}\theta$ describe the coupling of the vortex to: (i) the applied superflow $\boldsymbol{v}$; (ii) the electric and magnetic fields via $(A_0, \boldsymbol{A})$; and (iii) the superconducting electrons via the Berry phase Γ. Variation of the coupling terms with respect to $\boldsymbol{r}_0$ gives the non-dissipative force

$$\boldsymbol{F}_{\text{nd}} = \frac{\rho_s h\omega}{2}\,(\boldsymbol{v} - \dot{\boldsymbol{r}}_0) \times \hat{\boldsymbol{z}} + \mathcal{O}\left(\xi_0^2/\lambda^2\right)$$

where ξ_0 is the zero-temperature coherence length, and λ is the London penetration depth. Our result for $\boldsymbol{F}_{\text{nd}}$ is identical to the result found by Ao and Thouless [3] in the case of a neutral superfluid, and which they argued would also be true for a charged superconductor. In this first calculation we have considered the case of a *charged* superconductor explicitly (within the context of BCS superconductivity) and found that the Berry phase generated in the BCS ground state *is* responsible for producing the Magnus force contribution to $\boldsymbol{F}_{\text{nd}}$ as argued by Ao and Thouless [3], and that $\boldsymbol{F}_{\text{nd}}$ is given by the NV result. We go on now to the second *independent* calculation of $\boldsymbol{F}_{\text{nd}}$.

Our starting point (again) is the Bogoliubov equation for the case where a line vortex with winding number $\omega = -1$ is present. We transform the Bogoliubov Hamiltonian using the unitary operator $U = \exp[\mathrm{i}\theta\sigma_3/2]$ to obtain $H_{\mathrm{Bog}} = \sigma_3[(\mathrm{i}\nabla - \sigma_3 v_s)^2/(2) - E_F] + \Delta_0\sigma_1$. Here the $\{\sigma_i\}$ are the 2×2 Pauli matrices; E_F is the Fermi energy; and $v_s = -(1/2)\nabla\theta - eA$ ($\hbar = m = c = 1$). *We make an eikonal approximation* [9] *for the Bogoliubov equation* eigenstates $\Phi = \exp[\mathrm{i}q\cdot r]\Phi'$, where $|q| = k_F$ and Φ' varies on a length scale $L \gg k_F^{-1}$. To first order in gradients, this gives $H_{\mathrm{Bog}} = \sigma_3[-q\cdot(\mathrm{i}\nabla - \sigma_3 v_s)] + \Delta_0\sigma_1$, from which we obtain the gauge-invariant second-quantized Lagrangian

$$\mathcal{L}(\hat{q}) = \Psi^\dagger\left[\mathrm{i}\,\partial_t + \sigma_3\left(\tfrac{1}{2}\partial_t - eA_0\right) + \sigma_3 q\cdot(\mathrm{i}\nabla - \sigma_3 v_s) - \Delta_0\sigma_1\right]\Psi.$$

We see that the eikonal approximation made for the eigenstates of H_{Bog} near the Fermi surface in terms of wavepackets with mean momentum $p_F\hat{q}$ has led to the separation *of the (3 + 1)-dimensional* NQP *dynamics into a collection of independent (1 + 1)-* dimensional subsystems labelled by directions along the Fermi surface $\hat{q}$ and which we will refer to as $\hat{q}$-channels. By construction, both positive- and negative-energy eigenstates (that is, above and below the Fermi surface) carry a mean momentum $p_F\hat{q}$. Positive-energy quasiparticles in this channel carry (mean) momentum $p_F\hat{q}$ (right-goers, $\psi_R^\dagger$), while positive-energy quasiholes have (mean) momentum $-p_F\hat{q}$ (left-goers, ψ_L), and spin *indices have been suppressed. The adjoint of the* NQP *field operator in this channel is* $\Psi_{\hat{q}}^\dagger(x) = (\psi_R^\dagger(x;\hat{q}),\ \psi_L(x;\hat{q}))$. The Noether current associated with the global phase transformation $\Psi_{\hat{q}} \to \exp[-\mathrm{i}\chi]\Psi_{\hat{q}}$ can be written in a pseudo-relativistic notation as $j^\mu = \bar{\Psi}\gamma^\mu\Psi$. Here $\mu = 0, 1$; $x^0 \equiv t$, $x^1 \equiv q\cdot x$; $\gamma^0 \equiv \sigma_1$, $\gamma^1 \equiv -\mathrm{i}\sigma_2$; and $\bar{\Psi} \equiv \Psi^\dagger\gamma^0$. One can then write the density of linear momentum (in the $\hat{q}$-channel) as $g_i(x;\hat{q}) = p_F\hat{q}_i j^0(x)$; and the associated stress tensor as $T_{ij}(x;\hat{q}) = p_F\hat{q}_i\hat{q}_j j^1(x)$. Taking the expectation value of these operators with respect to the $\hat{q}$-channel ground state $|\mathrm{vac}\rangle_{\hat{q}}$, and summing over all $\hat{q}$-channels, gives the ground-state density of linear momentum $g_i(x)$ in the condensate

$$g_i(x) = k_F^3\sum_\alpha\int\frac{\mathrm{d}\hat{q}}{4\pi^2}\,\hat{q}_i\langle\mathrm{vac}|j^0(x;\hat{q})|\mathrm{vac}\rangle_{\hat{q}} \tag{5}$$

and its associated stress tensor given by

$$T_{ij}(x) = k_F^3\sum_\alpha\int\frac{\mathrm{d}\hat{q}}{4\pi^2}\,\hat{q}_i\hat{q}_j\,\langle\mathrm{vac}|j^1(x;\hat{q})|\mathrm{vac}\rangle_{\hat{q}}. \tag{6}$$

Here α is the spin index ($\pm$). The continuity equation for the condensate linear momentum is then

$$\partial_t g_i + \partial_j T_{ij} = k_F^3\sum_\alpha\int\frac{\mathrm{d}\hat{q}}{4\pi^2}\,\hat{q}_i\langle\mathrm{vac}|\partial_\mu j^\mu|\mathrm{vac}\rangle_{\hat{q}}. \tag{7}$$

The matrix element appearing in equation (7) does not vanish, signalling that the condensate linear momentum is not conserved (not surprising since the condensate is not isolated). We will see shortly that, together with the expected source terms due to gradients in the chemical *potential, and from the electric and magnetic fields; there will also be a source term whose* origin is the vortex topology and which keeps track of the rate at which linear momentum is disappearing into the vortex. This topological term will thus give F_{nd} in this second approach. Details of the calculation of $M = \langle\mathrm{vac}|\partial_\mu j^\mu|\mathrm{vac}\rangle_{\hat{q}}$ are given in [4]. The result is $M = (\epsilon^{\mu\nu}\tilde{F}_{\mu\nu})/4\pi$, where $\epsilon^{01} = 1$; $\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$; and $\tilde{A}_0 = eA_0 - (1/2)\,\partial_t\theta$; $\tilde{A}_1 = q\cdot v_s$. *Inserting this result for M into equation (7) gives*

$$\partial_t g_i + \partial_j T_{ij} = C_0\left\{-\frac{\hbar}{2}\,[\partial_0, \partial_i\,]\,\theta + eE_i\right\}. \tag{8}$$

Here $C_0 = k_F^3/3\pi^2$ is the particle density in the normal phase for the case where the chemical potential equals the Fermi energy; and $\hbar$ has been restored. The first term on the RHS of equation (8) is non-vanishing due to the non-trivial vortex topology. The local expression of this topology, appropriate for a vortex with winding number ω, is $[\partial_x, \partial_y]\phi = 2\pi\omega\,\delta(x - x_0)\,\delta(y - y_0)$, where ϕ is the gap phase and $r_0 = (x_0, y_0)$ is the position of the vortex. The LHS of equation (8) can also be evaluated using equations (5) and (6) along with the result [4] $\langle \mathrm{vac}|j^\mu|\mathrm{vac}\rangle_{\tilde{q}} = \epsilon^{\mu\nu}\tilde{A}_\nu/2\pi$. The results are $g_i = C_0\,(v_s)_i$; $T_{ij} = C_0\,((\hbar/2)\partial_t\theta - eA_0)\,\delta_{ij}$. Making use of these results in equation (8), together with the Josephson equation $(\hbar\,\partial_t\phi)/2 = -\mu_0$, where μ_0 is the chemical potential in the vortex rest frame which can be written as $\mu_0 = \mu + v_s^2/2 + eA_0$ (μ is the chemical potential in the lattice frame and $m = 1$) gives (finally)

$$\frac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} = -\nabla\mu + e\boldsymbol{E} + e\boldsymbol{v}_s \times \boldsymbol{B} - \frac{h\omega}{2}(\boldsymbol{v}_s - \dot{\boldsymbol{r}}_0) \times \hat{\boldsymbol{z}}\,\delta^2(\boldsymbol{r} - \boldsymbol{r}_0). \tag{9}$$

We see that the continuity equation for the condensate linear momentum has yielded the acceleration equation for the superflow. We find the expected source terms related to the hydrodynamic pressure ($\nabla P = \rho_s\,\nabla\mu$), and the electric and magnetic fields. We also see that linear momentum is disappearing from the condensate into the vortex at $r_0(t)$ at the rate $(\rho_s h\omega/2)(v_s - \dot{r}_0) \times \hat{z}$ per unit length, so

$$\boldsymbol{F}_{\mathrm{nd}} = \frac{\rho_s h\omega}{2}(\boldsymbol{v}_s - \dot{\boldsymbol{r}}_0) \times \hat{\boldsymbol{z}}$$

in agreement with the Berry phase calculation. Our result is also consistent with the calculation of NV in [1]. These authors showed that the first three terms in equation (9) lead to a flux of linear momentum in towards the vortex at a rate $(\rho_s h\omega/2)(v_s - \dot{r}_0) \times \hat{z}$ which is exactly the rate at which we find it appearing on the vortex, indicating that linear momentum is conserved in the combined condensate–vortex system.

In this work we have provided two *independent* microscopic calculations of the non-dissipative force F_{nd} acting on a line vortex in a type-II superconductor at $T = 0$. Both calculations yield the NV form for this force $F_{\mathrm{nd}} = (\rho_s h\omega/2)(v - \dot{r}_0) \times \hat{z}$. The first calculation (inspired by earlier work of Ao and Thouless which determined F_{nd} via a Berry phase analysis appropriate for a neutral superfluid, and which they argued would also be valid for a charged superconductor) shows that the arguments of Ao and Thouless are fully borne out in the context of the BCS model for a *charged* superconductor. The second calculation (which does not rely on Berry phases) examines the flow of linear momentum in the condensate. The continuity equation for this linear momentum is shown to: (i) yield the acceleration equation for the superflow; and (ii) to contain a sink term indicating the disappearance of linear momentum into the vortex. F_{nd} follows in this second approach from the rate of momentum loss to the vortex. The result obtained is the NV result, and the Magnus force (contribution to F_{nd}) is seen to be a consequence of the vortex topology and motion, exactly as it is in classical hydrodynamics.

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Note added in proof. Two preprints have appeared since this work was completed (Stone; Aitchison *et al* [10]) which also find a gauge-invariant contribution to the hydrodynamic action that is first order in time derivatives of the gap phase.

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[2] We assume a line vortex with its axis along $\hat{z}$; ρ_s is the superconducting electron density at $T = 0$ far from the vortex; h is Planck's constant; $\omega = \pm 1$ is the vortex winding number and corresponds to a single flux quantum $\phi_0 = hc/2e$ threading the vortex; v_s is the velocity of an applied supercurrent with respect to the lattice rest frame; m is the electron mass; $\mathcal{K} = h\omega/2m$ is the circulation of the condensate near the vortex core; and v_L is the translational velocity of the vortex with respect to the lattice.

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